Guide to Proofs on Sets

General Principles

To formalize your intuition about sets and how they behave – and to build up better predictions for how sets will interact with one another – you'll want to shift your thinking from a holistic "$A \cup B$ represents the set you get when you combine everything from A and B together" to a more precise "$x \in A \cup B$ if and only if $x \in A$ or $x \in B$." That change in perspective – from the properties of the set as a whole to properties of the individual elements of those sets – will be a key theme throughout this course. It's not necessarily the most natural perspective to adopt, but once you've learned to think about things this way you'll get a much deeper understanding for how sets behave.

The rest of this handout explores how to think about things this way, as well as how to use that perspective to write formal proofs.

A First Running Example

In the upcoming sections, we're going to see how to reason rigorously about sets and set theory. Rather than doing that in the abstract, we'll focus on a specific, concrete example.

Consider the following theorem:

> ***Theorem:*** For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Although there are a ton of variables here, this result isn't as scary as it might look. Before moving on, take a minute to think through what's going on here. Draw some pictures. Try out some examples. Even though you might not have an idea of where this proof is going to go, you can still at the very least set up the first sentence. As we saw in lecture, there are a number of little mini "proof templates" that you can use to focus your efforts. Here, we're trying to prove a universally-quantified statement ("*for all sets … if …then")*, and as you saw in class, there's a nice template for starting this one off:

> ***Theorem:*** For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.
>
> ***Proof:*** Pick arbitrary sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We want to show that $A \subseteq D \cup E$. [the rest of the proof goes here.]

In other words, we're beginning with a ton of little assumptions, and we have a single goal that we need to prove (namely, $A \subseteq D \cup E$). So now the question is how we go about doing that.

Reasoning About Subsets

The subset-of relation $\subseteq$ is one of the most fundamental relations we'll explore in set theory. As a reminder, formally speaking, we say that $S \subseteq T$ if every element of S is also an element of T. If you'll notice, the statement "every element of S is also an element of T" is a universal statement – it says that for each object of some type ("every element of S") has some other property ("is also an element of T.") And as you saw on Wednesday, there's a nice technique for proving universal statements that involves making arbitrary choices.

Putting this together, we have the following:

☞ **To prove $S \subseteq T$, pick an arbitrary $x \in S$, then prove that $x \in T$.** ☜

Using this template, we can continue the proof that we set up on the previous page. When we left off, we said we needed to prove $A \subseteq D \cup E$. And hey! We just developed a template for that. Let's use it!

Theorem: For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Pick arbitrary sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We want to show that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$. [*the rest of the proof goes here.*]

An important detail here: ***this proof introduces a new variable x.*** The statement of the theorem purely relates A, B, C, D, and E to one another. It says nothing whatsoever about anything named x. Although there is no variable x in the original theorem, *proving* that theorem requires us to reason about elements of the sets. How do we proceed from here? It's not immediately clear, but we can use some of the information we have. For example, we know that $A \subseteq B \cup C$, and we know that $x \in A$. We can combine these pieces of information together given the following principle:

☞ **If you know $x \in S$ and $S \subseteq T$, you can conclude $x \in T$.** ☜

This follows from how subsets are defined. If $S \subseteq T$, then every element of S is an element of T, and so in particular because x is an element of S, we can say that x is an element of T. We can put that into practice here:

Theorem: For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Pick arbitrary sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We want to show that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. [*the rest of the proof goes here.*]

A detail to point out before we move on: notice that the way that we interact with the $\subseteq$ relation in a proof differs based on whether we are ***proving*** that one set is a subset of another or whether we are ***using*** the fact that one set is a subset of another. That will be unifying theme throughout the entire quarter, and you'll see this come up in the rest of this handout. In the first paragraph, we set up a ***proof*** that $A \subseteq D \cup E$ by picking an arbitrary $x \in A$. In the second, we ***used*** the fact that $A \subseteq B \cup C$ to conclude that $x \in B \cup C$. ***Proving*** that one set is a subset of another introduces a new variable; ***using*** the fact that one set is a subset of the other lets us conclude new things about existing variables.

Reasoning About Set Combinations

You probably have a good intuition for unions, intersections, and the like from your lived experience. The union of the set of all your TAs and your classmates represents the set of people you're mostly like to interact with in a given course. The intersection of the set of people you admire and the set of people who admire you represents the set of people you probably should consider becoming friends with. And so on.

But in the elemental theory of sets, we have to ask – what exactly makes up the sets $S \cup T$, $S \cap T$, $S \Delta T$, etc.? After all, sets are formally defined by their elements. And for that, we need these definitions:

$S \cup T = \{ x \mid x \in S \text{ or } x \in T \text{ (or both)} \}$ $S \cap T = \{ x \mid x \in S \text{ and } x \in T \}$

$S - T = \{ x \mid x \in S \text{ and } x \notin T \}$ $S \Delta T = \{ x \mid \text{either } x \in S \text{ and } x \notin T\text{, or } x \notin S \text{ and } x \in T \}$

These are the definitions of these terms. It's good to know these definitions when you're thinking about how these sets operate. But in the context of proofwriting, you'll likely need to *use* these definitions in the following way:

☞ **If you know $x \in S \cup T$, you can conclude $x \in S$ or $x \in T$.**

If you know $x \in S \cap T$, you can conclude $x \in S$ and $x \in T$.

If you know $x \in S - T$, you can conclude $x \in S$ and $x \notin T$.

If you know $x \in S \Delta T$, you can conclude either $x \in S$ and $x \notin T$, or $x \notin S$ and $x \in T$. ☜

Let's jump back to the proof we're working through. We know that $x \in B \cup C$. Given what we just saw above, we can use that to conclude that $x \in B$ or $x \in C$. Let's use that to our advantage:

Theorem: For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Pick arbitrary sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We want to show that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. This in turn tells us that $x \in B$ or $x \in C$. [the rest of the proof goes here.]

We're making some progress here, because this lets us use some of the facts from our proof setup that we haven't touched yet. Specifically, we've been holding onto the fact that $B \subseteq D$ and that $C \subseteq E$, and here we're confronted with the fact that either $x \in B$ or $x \in C$. Using what we saw in the previous section about subsets, that means that we can potentially make a lot more progress here. The challenge is that we can't say for certain whether $x \in B$ or $x \in C$ – that might depend on x, B, and C. But that's not a problem – that's the sort of thing a proof by cases was meant for!

Here's how we might continue from the previous section using both a proof by cases and our knowledge of how B, C, D, and E relate:

Theorem: For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Pick arbitrary sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We want to show that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. This in turn tells us that $x \in B$ or $x \in C$. We will therefore proceed by cases:

Case 1: $x \in B$. Then since $x \in B$ and $B \subseteq D$, we see that $x \in D$.

Case 2: $x \in C$. Then since $x \in C$ and $C \subseteq E$, we see that $x \in E$.

[*the rest of the proof goes here.*]

This is looking a lot better.

Our ultimate goal is to prove that $x \in D \cup E$. And based on where we are now, it seems like that goal is in sight! We know that $x \in D$ or that $x \in E$. And intuitively, that seems like that should be enough to conclude that $x \in D \cup E$, since, after all, $D \cup E$ is what you get when you take all the elements of D and all the elements of E and combine them together.

Above, we saw how you could use the formal definitions of the set combination operators to go from knowledge that $x \in S \cup T$ to the conclusion that $x \in S$ or $x \in T$. In other words, if we already happen to know that an object is an element of a set union, we can use that to learn something about how that object connects with the individual sets that make up that union. But what about the other direction? What do we have to do to show that an object is an element of the union of two sets? For that, we can use this handy table:

☞ **To prove $x \in S \cup T$, prove that $x \in S$ or that $x \in T$.**

To prove $x \in S \cap T$, prove that $x \in S$ and $x \in T$.

To prove $x \in S - T$, prove that $x \in S$ and $x \notin T$.

To prove that $x \in S \Delta T$, prove that $x \in S$ and $x \notin T$, or that $x \notin S$ and $x \in T$. ☜

With this in mind, we can finish our proof! In each case, we learn that x belongs to one of the D or E, and so we can conclude that it always belongs to $D \cup E$.

Theorem: For all sets A, B, C, D, and E, if $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Pick arbitrary sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We want to show that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. This in turn tells us that $x \in B$ or $x \in C$. We will therefore proceed by cases:

Case 1: $x \in B$. Then since $x \in B$ and $B \subseteq D$, we see that $x \in D$.

Case 2: $x \in C$. Then since $x \in C$ and $C \subseteq E$, we see that $x \in E$.

Collectively, these cases show that $x \in D$ or that $x \in E$. Therefore, we see that $x \in D \cup E$, as required. ■

And that's a wrap! Now, look back over this proof. Notice that it is focused on a single element x, went with x on a magical journey, and ended up reaching our desired conclusion.

Reasoning About Set Equality

What does it mean for two sets to be equal? This is addressed by the fancy-sounding *axiom of extensionality*, a term you are totally welcome to toss around at cocktail parties, which says the following:

☞ **Two sets S and T are equal ($S = T$) if $S \subseteq T$ and $T \subseteq S$** ☜

This definition of set equality lets you make the following conclusions in the case where you know two sets are equal to one another:

☞ **If $S = T$ and $x \in S$, you can conclude that $x \in T$.**

If $S = T$ and $x \notin S$, you can conclude that $x \notin T$. ☜

The definition of set equality gives us the following route for proving two sets are equal:

☞ **To prove that $S = T$, prove that $S \subseteq T$ and $T \subseteq S$.** ☜

This approach for proving that two sets are equal is sometimes called a ***proof by double inclusion***, though we generally won't refer to it by that name. You are welcome to toss that around at cocktail parties as well, though, if you so choose.

Another consequence of this theorem is the following conclusion that you can also draw from two sets being equal to one another:

☞ **If $S = T$, you can conclude that $S \subseteq T$ and $T \subseteq S$.** ☜

This comes up every now and then, though it's much more common to use the opposite direction of this theorem to prove that $S = T$ via $S \subseteq T$ and $T \subseteq S$.

Reasoning About Power Sets

First, let's recap the formal definition of the power set. The power set of a set S is the set of all subsets of S:

$$\wp(S) = \{ T \mid T \subseteq S \}$$

If you haven't already done so, take a minute to read over that set-builder notation and to see if you can convince yourself why it says symbolically what we described in plain English right above it.

The above definition is wonderfully useful. For example, if you want to show that an object belongs to $\wp(S)$, you need to show that that object obeys the set-builder notation. Specifically:

☞ **To prove that $T \in \wp(S)$, prove that $T \subseteq S$.** ☜

You can also run this definition the other way:

☞ **If you know $T \in \wp(S)$, you can conclude $T \subseteq S$.** ☜